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A VARIATIONAL PRINCIPLE IN CALCULATING THE LONG-TERM STRENGTH OF STRUCTURES†

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A variational principle of mixed type is proposed to determine the stress–strain state of a non-linear viscoelastic body, on the assumption that the processes of creep and damage are interrelated and taking geometrical non-linearity into account. A special feature of the functional is its formulation in terms of accelerations, which enables the system of Euler equations to be expanded by adding, besides the equilibrium equations, an equation of vulnerability to damage. The functional has been tested in connection with the problem of the long-term strength of a cylindrical shell under internal pressure.

In connection with problems in designing structures for long-term strength, we shall treat failure as a diffuse process and assume that the body manifests hereditary properties during failure. We shall assume that the body is viscoelastic, taking into consideration that the physical equations involve a parameter ω defining damage accumulation. Assuming that the volume deformation is elastic, we take the physical equations to be

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E} g_{ij}\sigma^{kl} g_{kl} + \frac{3}{2}\int_0^t K(t-\tau,\omega) f(I,\omega)(\sigma_{ij} - \frac{1}{3} g_{ij}\sigma^{kl} g_{kl}) d\tau \tag{1}$$

$$I^2 = \frac{3}{2}\left(\sigma^{ij} - \frac{1}{3} g^{ij}\sigma^{kl} g_{kl}\right)\left(\sigma_{ij} - \frac{1}{3} g_{ij}\sigma^{pq} g_{pq}\right)$$

where, besides the standard notation ,we let f denote a function of non-linearity (if $f = 1$ and $\partial k/\partial\omega = 0$, we obtain a linear viscoelastic body), g_{ij} are the components of the metric tensor and I is the first invariant of the stress tensor.

To define the damage parameter ω, we use a kinetic equation of vulnerability to damage [or, briefly, the damage equation] which, guided by analogy with (1), we take as follows [2]:

$$\omega = \int_0^t c(t-\tau)\varphi(I,\omega) d\tau \tag{2}$$

where φ is damage function and $c(t)$ is the damage kernel. It is obvious from these relations that, if the stresses are constant with time, damage may accumulate, in keeping with our initial premises.

This non-linear problem is further complicated by allowance for large displacements, which is characteristic for problems in viscoelasticity. To solve such problems one must resort to approximate methods, such as the variational method.

One possible functional is the following

$$J = \int_V \left\{ \frac{1}{2}\dot{\sigma}^{ij}(\ddot{u}_{i,j} + \ddot{u}_{j,i} + \ddot{u}^k_{,i} u_{k,j} + 2\dot{u}^k_{,i}\dot{u}_{k,j} + u^k_{,i}\ddot{u}_{k,j}) + \sigma^{ij} u^k_{,j}\ddot{u}_{k,i} - \right.$$

$$-\dot{\sigma}^{ij}\left(\frac{1+\nu}{E}\ddot{\sigma}_{ij} - \frac{\nu}{E} g_{ij}\ddot{\sigma}^{kl} g_{kl}\right) - \frac{3}{2}\dot{\sigma}^{ij}\left[\int_0^t K''_{tt}(t-\tau,\omega) f(I,\omega)\left(\sigma_{ij} - \frac{1}{3} g_{ij}\sigma^{kl} g_{kl}\right) d\tau + \right.$$

$$+[K(0,\omega) f(I,\omega)]'_\omega \dot{\omega}\left(\sigma_{ij} - \frac{1}{3} g_{ij}\sigma^{kl} g_{kl}\right) + \frac{1}{2} K(0,\omega) f'_I \frac{\partial I}{\partial\sigma^{pq}}\dot{\sigma}^{pq} \times$$

$$\times\left(\sigma_{ij} - \frac{1}{3} g_{ij}\sigma^{kl} g_{kl}\right) + \frac{1}{2} K(0,\omega) f(I,\omega)\left(\dot{\sigma}_{ij} - \frac{1}{3} g_{ij}\dot{\sigma}^{kl} g_{kl}\right) +$$

$$\left. + K'_t(0,\omega) f(I,\omega)\left(\sigma_{ij} - \frac{1}{3} g_{ij}\sigma^{kl} g_{kl}\right)\right] -$$

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$$-\left(\ddot{\omega}-\int_0^t C''_{tt}(t-\tau)\varphi(I,\omega)d\tau-C'_t(0)\varphi(I,\omega)-C(0)\frac{\partial\varphi}{\partial\omega}\dot{\omega}\frac{1}{2}\right)\dot{\omega}C^{-1}(0)\times$$

$$\left.\times\left(\frac{\partial\varphi}{\partial I}\right)^{-1}I[K(0,\omega)f(I,\omega)]'_{\omega}\right\}dV-\int_{S_\sigma}\dot{\bar{T}}^i\ddot{u}_i dS+\int_{S_u}\dot{T}^i(\ddot{\bar{u}}_i-\ddot{i}_i)dS \tag{3}$$

where V is the volume of the body, u_i are the components of the displacement vector, a comma denotes covariant differentiation, a prime means differentiation with respect to the relevant argument and a dot means differentiation with respect to t. The formula for the functional (3) assumes that the problem is formulated for the case in which the boundary conditions are separated namely we mean: by S_σ the part of the surface of the body with volume V on which the stresses $\bar{T}^k$ are given and have the following form [1]

$$\bar{T}^k=\sigma^{ij}(\delta_j^k+u_{,j}^k)n_i$$

where n_i are the components of the normal vector and δ is the Kronecker delta; on the remaining part of the surface the displacement vector u_i is given. The quantities to be varied are $\ddot{u}_i$, $\dot{\sigma}^{ij}$ and ω. It is assumed throughout that the variation operator δ acts on the varied quantities only. The following principle is proposed: those functions that give the functional (3) a stationary value under the variation conditions just described also determine the non-linear behaviour of the viscoelastic body with allowance for damage, i.e. they satisfy the non-linear equilibrium equations in a Cartesian system of coordinates

$$[\sigma^{ij}(\delta_j^k+u_{,j}^k)]_{,i}=0 \tag{4}$$

determine the displacements in terms of the stresses

$$\frac{1}{2}(u_{i,j}+u_{j,i}+u_{,i}^k u_{k,j})=\frac{1+\nu}{E}\sigma_{ij}-\frac{\nu}{E}g_{ij}\sigma^{kl}g_{kl}+$$

$$+\frac{3}{2}\int_0^t K(t-\tau,\omega)f(I,\omega)\left(\sigma_{ij}-\frac{1}{3}g_{ij}\sigma^{kl}g_{kl}\right)d\tau \tag{5}$$

satisfy the boundary conditions

$$T^i=\bar{T}^i,\quad x\in S_\sigma;\quad u_i=\bar{u}_i,\quad x\in S_u \tag{6}$$

and at the same time the kinetic damage equation (2) is satisfied.

To prove this assertion, let us determine the first variation of the functional (3), given the variation conditions. We obtain

$$\delta J=\int_V\left\{\delta\dot{\sigma}^{ij}\tfrac{1}{2}(\ddot{u}_{i,j}+\ddot{u}_{j,i}+\ddot{u}_{,i}^k u_{k,j}+2\dot{u}_{,i}^k\dot{u}_{k,j}+u_{,i}^k\ddot{u}_{k,j})+\sigma^{ij}\dot{u}_{,j}^k\delta\ddot{u}_{k,i}+\right.$$

$$+\dot{\sigma}^{ij}(\delta\ddot{u}_{i,j}+u_{k,j}\delta\ddot{u}_{,i}^k)-\delta\dot{\sigma}^{ij}(\tfrac{3}{2})\left[\int_0^t K''_{tt}(t-\tau,\omega)f(I,\omega)(\sigma_{ij}-\tfrac{1}{3}g_{ij}\sigma^{kl}g_{kl})d\tau+\right.$$

$$+[K(0,\omega)f(I,\omega)]'_{\omega}\dot{\omega}(\sigma_{ij}-\tfrac{1}{3}g_{ij}\sigma^{kl}g_{kl})+K(0,\omega)f'_I\frac{\partial I}{\partial\sigma^{pq}}\dot{\sigma}^{pq}\left(\sigma_{ij}-\frac{1}{3}g_{ij}\sigma^{kl}g_{kl}\right)+$$

$$\left.+K(0,\omega)f(I,\omega)\left(\dot{\sigma}_{ij}-\frac{1}{3}g_{ij}\dot{\sigma}^{kl}g_{kl}\right)+K'_t(0,\omega)f(I,\omega)\left(\sigma_{ij}-\frac{1}{3}g_{ij}\sigma^{kl}g_{kl}\right)\right]-$$

$$-\left(\ddot{\omega}-\int_0^t C''_{tt}(t-\tau)\varphi(I,\omega)d\tau-C'_t(0)\varphi(I,\omega)-C(0)\varphi'_{\omega}\dot{\omega}-\dot{\sigma}^{ij}C(0)\varphi'_I\frac{\partial I}{\partial\sigma^{ij}}\right)\times$$

$$\left.\times\delta\dot{\omega}\left(\frac{\partial\varphi}{\partial I}\right)^{-1}C^{-1}(0)I[K(0,\omega)f(I,\omega)]'_{\omega}\right\}dV-\int_{S_\sigma}\dot{\bar{T}}^i\delta\ddot{u}_i dS+\int_{S_u}[\delta\dot{T}^i(\ddot{\bar{u}}_i-\ddot{u}_i)-\dot{T}^i\delta\ddot{u}_i]dS$$

(we have used the Gauss–Ostrogradskii theorem). The fundamental lemma of variational calculus and the relation

$$\sigma_{ij}-\frac{1}{3}g_{ij}\sigma^{kl}g_{kl}=\frac{2}{3}I\frac{\partial I}{\partial\sigma^{ij}} \tag{7}$$

yield Euler's equation for the functional (3) in Cartesian coordinates

$$\begin{gathered}\left\{\frac{1}{2}(u_{i,j}+u_{j,i}+u^k_{,i}u_{k,j})-\frac{1+\nu}{E}\sigma_{ij}+\frac{\nu}{E}g_{ij}\sigma^{kl}g_{kl}-\frac{3}{2}\int_0^t K(t-\tau,\omega)f(I,\omega)\times\right.\\ \left.\times\left(\sigma_{ij}-\frac{1}{3}g_{ij}\sigma^{kl}g_{kl}\right)d\tau\right\}^{\cdot\cdot}=0\\ \left\{[\sigma^{ik}(\delta^k_j+u^k_{,j})]_{,i}\right\}^{\cdot}=0,\quad \left\{\omega-\int_0^t C(t-\tau)\varphi(\omega,I)d\tau\right\}^{\cdot\cdot}=0\\ \{T^i-\bar{T}^i\}^{\cdot}=0,\quad x\in S_\sigma;\quad \{u_i-\bar{u}_i\}^{\cdot\cdot}=0,\quad x\in S_u\end{gathered} \tag{8}$$

We have thus obtained a system of defining equations in differential form. The principle may thus be considered as proved.

When a stationary value of the functional (3) is determined by numerical means, the problem reduces to solving a system of integrodifferential equations.

It follows from the form of system (8) that it may be integrated with respect to time. To that end we impose initial conditions. We first solve the non-linear viscoelastic problem with the above boundary conditions but with the kernel $K(t-\tau, 0)$. As the initial condition for system (8) we take a solution of this viscoelastic problem with this kernel. In addition, we must also add initial conditions for the damage parameter

$$\omega=0,\quad \dot{\omega}=C(0)\varphi(0,I(0))\quad \text{for}\quad t=0$$

It is obvious that system (8) with this initial condition is integrable. Integration yields a system of equations identical with (8), but without the braces and the dots over them removed, i.e. Eqs (2) and (4)–(6). This proves the principle.

As regards the choice of initial conditions, note that in order to integrate system (7) we have to know not only the stress–strain state at time $t = 0$ but also the rate at which this state changes at the starting time.

This principle has some special features. The integrand of the functional contains a term involving the factor $(\partial\varphi/\partial I)^{-1}$. If $\partial\varphi/\partial I = 0$, the integral has a singularity. In that case we conclude that φ does not depend on the stressed state. Then ω may be determined from the kinetic equation, independently of the solution of the whole problem, and the variational principle is no longer needed. For that reason the case $\partial\varphi/\partial I = 0$ will not be considered here.

There are two reasons for choosing a functional of this type. First, the resulting system of Euler equations (after differentiation of the second equation of system (8) with respect to t) is quasilinear in $\ddot{u}_i$, $\ddot{\sigma}^{ij}$, $\ddot{\omega}$ and well-tried procedures for the computerized solution of the corresponding Cauchy problem exist. Second, unlike other functionals [3], the kinetic damage equation is in this case an Euler equation, not an additional condition, so that simple functions are adequate to approximate both the solution of the non-linear equilibrium equations and the solution of the kinetic damage equation.

There are many functionals analogous to (3) (see [4–6]) when the Euler equation also includes the kinetic damage equation. The special feature of the functional (3) is that it is designed for non-linearly viscoelastic bodies ($f \neq 1$), there being no need to use a combination of the components of the stress tensor—the treatment of the latter in specific examples produces cumbersome expressions.

Let us illustrate the application of our principle with a simple example: the behaviour of a long cylindrical shell of length L, thickness $2h$ and internal surface radius R. Suppose that the shell is subjected to internal pressure of intensity $q(t)$, uniformly distributed over the surface.

The physical equations of viscoelasticity for the one-dimensional case may be taken as ($K(t)$ = const)

$$\varepsilon=\frac{\sigma}{E}+A\int_0^t\frac{\sigma^n}{(1-\omega)^m}d\tau \tag{9}$$

and the kinetic damage equation may be written as ($c(t)$ = const)

$$\omega=B\int_0^t\sigma^f\frac{1}{(1-\omega)^g}d\tau \tag{10}$$

where A and B are mechanical parameters.

The one-dimensional equations (9) and (10) are applicable in this case provided the shell is sufficiently long, so that end effects may be ignored, and because the problem is axisymmetric, as follows from the loading conditions. Utilizing the smallness of the deflection W, we can write the functional as

$$J=\int_0^L\int_{-h}^{h}\left\{\dot\sigma\frac{\ddot W}{R}-\frac{1}{E}\dot\sigma\ddot\sigma-\frac{1}{2}\frac{nA\sigma^{n-1}}{(1-\omega)^m}\dot\sigma^2-A(1-\omega)^{-m-1}m\sigma^n\dot\omega\dot\sigma+\right.$$
$$\left.+\left[\ddot\omega-\frac{1}{2}B\sigma^f g\dot\omega(1-\omega)^{-g-1}\right]\dot\omega\frac{A}{B}\frac{m}{f}\sigma^{-n-f+1}(1-\omega)^{-1+g-m}-\dot q\ddot W\frac{1}{2h}\right\}dzdx \tag{11}$$

where x is the longitudinal coordinate.

Starting from the expected pattern of the shell's behaviour, we shall assume that all the unknown quantities are independent of the coordinates

$$\sigma=N_0/(2h),\qquad W=W_0,\qquad \omega=\omega_0 \tag{12}$$

where N_0, W_0 and ω_0 are unknown functions of time and load. Taking the approximate form (11) of the functional into account, we obtain the following function of the approximation coefficients

$$J=L\times 2h\left\{\ddot W_0\frac{1}{R}\frac{\dot N_0}{2h}-\frac{1}{E}\frac{\ddot N_0}{2h}\frac{\dot N_0}{2h}-\left(\frac{N_0}{2h}\right)^{n-1}\frac{n}{2}A(1-\omega_0)^{-m}\left(\frac{\dot N_0}{2h}\right)^2-A(1-\omega_0)^{-m-1}m\times\right.$$
$$\times\left(\frac{N_0}{2h}\right)^n\dot\omega_0\frac{\dot N_0}{2h}+\left[\ddot\omega_0-\frac{1}{2}B\left(\frac{N_0}{2h}\right)^f g\dot\omega_0(1-\omega_0)^{-g-1}\right]\dot\omega_0\frac{A}{B}\frac{m}{f}(1-\omega_0)^{-m-1+g}\times$$
$$\left.\times\left(\frac{N_0}{2h}\right)^{n-f+1}-\dot q\frac{\ddot W}{2h}\right\}$$

Henceforth the zero subscript will be omitted. Since the functions being varied are $\dot N$, $\ddot W$, $\dot\omega$, the condition for the functional to be stationary is determined from the following system

$$\frac{\partial J}{\partial\ddot W}=\dot N\frac{1}{R}-\dot q=0$$
$$\frac{\partial J}{\partial\dot N}=\frac{1}{R}\ddot W-\frac{1}{E}\frac{1}{2h}\ddot N-\frac{An}{(1-\omega)^m}\left(\frac{N}{2h}\right)^{n-1}\dot N\frac{1}{2h}-\frac{A}{(1-\omega)^{m+1}}m\left(\frac{N}{2h}\right)^n\dot\omega=0 \tag{13}$$
$$\frac{\partial I}{\partial\dot\omega}=-A\frac{1}{(1-\omega)^{m+1}}m\frac{\dot N}{2h}\left(\frac{N}{2h}\right)^n+\left[\ddot\omega-B\left(\frac{N}{2h}\right)^f g\dot\omega(1-\omega)^{-g-1}\right]\times$$
$$\times\frac{A}{B}\frac{m}{f}\left(\frac{N}{2h}\right)^{n-f+1}(1-\omega)^{-m-1+g}=0$$

To solve this system of equations, we impose initial conditions. Based on our variational principle, we may assume that they are determined by the solution of our specific problem, but with the following physical equation, which differs from (9)

$$\varepsilon=\frac{\sigma}{E}+A\int_0^t\sigma^n d\tau$$

It is obvious that this problem must be solved on the same assumptions as underlie the construction of the functional (11), using an analogous method, e.g. Reisner's variational principle for viscoelastic bodies [1]. Thus, the functional has the form

$$J=\int_0^L\int_{-h}^{h}\left\{\sigma\frac{W}{R}-\frac{1}{2E}\sigma^2-A\sigma\int_0^t\sigma^n d\tau-q\frac{W}{2h}\right\}dxdz$$

where the varied quantities are W and σ, but the integral is not varied.

After applying the approximation (12), we express the defining system as follows:

$$N = q - R, \quad \frac{W}{R} = \frac{1}{E}\frac{N}{2h} + A\int_0^t N^n d\tau \frac{1}{(2h)^n} \tag{14}$$

Thus, system (14) determines the initial conditions for the solution of system (13): the values of W, $\dot{W}$, N, $\dot{N}$ at $t = 0$ are determined from (14). Further initial conditions are

$$t = 0; \quad \omega = 0, \quad \dot{\omega} = B\left[\frac{N(0)}{2h}\right]^f \tag{15}$$

Thus, system (13) must be solved with initial conditions (15) and the conditions determined from (14).

We differentiate the first equation of system (13) with respect to t. This yields a representation $\dot{N} = R\dot{q}$, which enables us to solve system (13) for the unknowns ($\dot{N}$, $\ddot{W}$, $\ddot{\omega}$) and to express the system in canonical form. The Cauchy problem thus obtained is generally solved by numerical means (until $\omega = 1$, when failure occurs).

In the problem under consideration here system (13) may be solved by analytical means. Indeed, taking the initial conditions and system (13) into account, we obtain

$$N = qR; \quad \omega = 1 - \left[1 - Bt(1+g)\left(q\frac{R}{2h}\right)^f\right]^{1/(1+g)}$$

$$\frac{W}{R} = \frac{1}{E}q\frac{R}{2h} + A\left(q\frac{R}{2h}\right)^n \int_0^t \left[1 - B\left(q\frac{R}{2h}\right)^f \tau(1+g)\right]^{-m/(1+g)} d\tau \tag{16}$$

These equations are derived on the assumption that $q(t) = \text{const}$. System (16) yields the critical failure time

$$t_* = \left[B\left(q\frac{R}{2h}\right)^f (1+g)\right]^{-1} \tag{17}$$

Note that t_* does not depend on the mechanical creep parameters A, m and n.

We will solve the problem taking into account the non-linearity of the deflection. The functional (3) takes the form

$$J = \int_0^L \int_{-h}^{h} \left\{\dot{\sigma}\left[\frac{\ddot{W}}{R} + \frac{\ddot{W}}{R}\frac{W}{R} + \left(\frac{\dot{W}}{R}\right)^2\right] + \sigma\frac{\dot{W}}{R}\frac{\ddot{W}}{R} - \frac{1}{E}\ddot{\sigma}\dot{\sigma} - \frac{1}{2}\frac{nA\sigma^{n-1}}{(1-\omega)^m}\dot{\sigma}^2 - A(1-\omega)^{-m-1}m\sigma^n\dot{\omega}\dot{\sigma} + \right.$$

$$\left. + \left[\ddot{\omega} - \frac{1}{2}B\sigma^f g\dot{\omega}(1-\omega)^{-g-1}\right]\dot{\omega}\frac{A}{B}\frac{m}{f}\sigma^{n-f+1}(1-\omega)^{-1+g-m} - \dot{q}\frac{\ddot{W}}{2h}\right\} dz dx \tag{18}$$

A stationary value of this functional may be determined by Ritz's method. Using the approximation (12), we obtain the following function of the approximation coordinates

$$J = L \times 2h\left\{\frac{1}{2h}\dot{N}\left[\frac{\ddot{W}}{R} + \frac{\ddot{W}}{R}\frac{W}{R} + \left(\frac{\dot{W}}{R}\right)^2\right] + N\frac{1}{2h}\frac{\dot{W}}{R}\frac{\ddot{W}}{R} - \frac{1}{E}\frac{1}{4h^2}\ddot{N}\dot{N} - \right.$$

$$-\left(\frac{1}{2h}N\right)^{n-1}\frac{n}{2}\frac{A}{(1-\omega)^m}\frac{1}{4h^2}\dot{N}^2 - A(1-\omega)^{-m-1}m\left(\frac{N}{2h}\right)^n\dot{\omega}\frac{\dot{N}}{2h} +$$

$$\left. + \left[\ddot{\omega} - \frac{1}{2}B\left(\frac{N}{2h}\right)^f g\dot{\omega}(1-\omega)^{-g-1}\right]\dot{\omega}\frac{A}{B}\frac{m}{f}(1-\omega)^{-m-1+g}\left(\frac{N}{2h}\right)^{n-f+1} - \dot{q}\ddot{W}\frac{1}{2h}\right\}$$

By analogy with (13), the defining system is

$$\frac{\partial J}{\partial W} = \dot{N}\frac{1}{R}\left(1 + \frac{W}{R}\right) + N\frac{1}{R^2}\dot{W} - \dot{q} = 0$$

$$\frac{\partial J}{\partial N} = \frac{1}{R}\left(\ddot{W} + \ddot{W}\frac{W}{R} + \frac{\dot{W}^2}{R}\right) - \frac{1}{E}\frac{1}{2h}\ddot{N} - \frac{An}{(1-\omega)^m}\left(\frac{N}{2h}\right)^{n-1}\dot{N}\frac{1}{2h} - \frac{A}{(1-\omega)^{m+1}}m\left(\frac{N}{2h}\right)^n\dot{\omega} = 0$$

$$\frac{\partial J}{\partial \dot{\omega}} = -A\frac{1}{(1-\omega)^{m+1}} m\frac{\dot{N}}{2h}\left(\frac{N}{2h}\right)^{n} + \left[\ddot{\omega} - B\left(\frac{N}{2h}\right)^{f} g\dot{\omega}(1-\omega)^{-g-1}\right]\times$$
$$\times\frac{A}{B}\frac{m}{f}\left(\frac{N}{2h}\right)^{n-f+1}(1-\omega)^{-m-1-g} = 0 \qquad (19)$$

To determine the initial conditions for system (19), we introduce the Reisner functional for the non-linear case

$$J = \int_{0}^{L}\int_{-h}^{h}\left\{\sigma\frac{W}{R}\left(1+\frac{W}{R}\frac{1}{2}\right) - \frac{1}{2E}\sigma^{2} - A\sigma\int_{0}^{t}\sigma^{n}d\tau - q\frac{W}{2h}\right\}dxdz$$

Using Ritz's method and the approximation (12), we determine a stationary value of this functional from the following system

$$N = qR\left(1+\frac{W}{R}\right)^{-1}, \quad \frac{W}{R}\left(1+\frac{1}{2}\frac{W}{R}\right) = \frac{1}{E}\frac{N}{2h} + A\left(\frac{1}{2h}\right)^{n}\int_{0}^{t}N^{n}d\tau \qquad (20)$$

Thus, the solution of the non-linear problem reduces to solving the system of equations (19) with initial condition (20).

Let us assume that $q(t)$ = const and that elastic deformation may be ignored. Then system (19) may be simplified and the second equation becomes

$$\frac{dc}{d\omega} = \frac{A}{B}\left(q\frac{R}{2h}\right)^{n-f}(1-\omega)^{g-m}(1+c)^{f-n-1}, \quad c = \frac{W}{R}$$

The kinetic equation remains unchanged.

These two equations yield the solution of our problem

$$t = \frac{1}{B}\left(q\frac{R}{2h}\right)^{-f}\int_{0}^{\omega}(1-\omega)^{g}\left\{\frac{A}{B}\left(q\frac{R}{2h}\right)^{n-f}\frac{n+2-f}{g-m+1}\left[1-(1-\omega)^{g-m+1}\right]+1\right\}^{f/(n+2-f)}d\omega \qquad (21)$$

Putting $\omega = 1$ in this solution, we obtain an expression for the critical failure time t_*. Comparing (21) at $\omega = 1$ with (17), we observe that in the non-linear case, if t_* exists, then, unlike the linear case, it depends on the mechanical creep parameters. In addition, whereas in the linear theory the critical time existed for all parameter values, in the non-linear theory the existence of t_*, i.e. the existence of the integral (21) at $\omega = 1$, needs special attention. Analysis of this problem indicates the need to allow for large deformations.

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